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SPACES HAVING A WEAKLY-INFINITE-DIMENSIONAL COMPACTIFICATION

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In this paper we introduce the concept of small-weak-infinite-dimensionality. We show that a separable metric space has a weakly-infinite-dimensional compact metric extension if and only if the space is small-weakly-infinite-dimensional.

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0. Introduction

In this paper all spaces are considered normal. We focus our attention on separable metric spaces. We follow the terminology used in [1, 2].

First we introduce some known concepts of infinite-dimension theory:

Definition 1. A space X is called *weakly-infinite-dimensional in the sense of Alexandrov* (*Smirnov*), abbreviated *A-w.i.d.* (*S-w.i.d.*), when for every sequence $\{(A_i, B_i)\}_{i=1}^{\infty}$ of pairs of disjoint closed sets in X there exist open sets V_i, $i = 1, 2, \ldots$, such that

$$A_i \subset V_i \subset \bar{V}_i \subset X - B_i$$

and $\bigcap_{i=1}^{\infty} \operatorname{Br} V_i = \emptyset$ ($\bigcap_{i=1}^{n} \operatorname{Br} V_i = \emptyset$ for some n).

Observe that, when X is compact, the notions A-w.i.d. and S-w.i.d. coincide. We then call the space X *w.i.d.*

Definition 2. A space X is called *countable dimensional*, abbreviated *c.d.*, when $X = \bigcup_{n=1}^{\infty} X_n$ with $\dim X_n < \infty$ for every n.

Definition 3. A space X satisfies $\operatorname{ind} X = -1$ iff $X = \emptyset$, $\operatorname{ind} X \leqslant \alpha$ iff for every point x in X and every nbd U of x, we can find a nbd V of x such that

$$x \in V \subset \bar{V} \subset U \quad \text{and} \quad \operatorname{ind} \operatorname{Br} V < \alpha.$$

Here α denotes some ordinal number. If for some ordinal number α, $\operatorname{ind} X \leqslant \alpha$ holds we say that X has *small transfinite dimension* or *ind*.

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We now start with the introduction of small-weak-infinite-dimensionality.

Definition 4. A collection of subsets $\mathscr{B}$ of a space X is called *inessential* when for every sequence $\{(B_i^1, B_i^2)\}_{i=1}^{\infty}$ of pairs of elements of $\mathscr{B}$ such that $\bar{B}_i^1 \cap \bar{B}_i^2 = \emptyset$ for every i, we can find open sets V_i for every i such that

$$B_i^1 \subset V_i \subset \bar{V}_i \subset X - B_i^2 \quad \text{and} \quad \bigcap_{i=1}^{n} \operatorname{Br} V_i = \emptyset$$

for some n.

If $\mathscr{B}$ is a collection of sets, then $\mathscr{B}^{\text{fin}}$ denotes the collection of all finite unions of elements of $\mathscr{B}$.

Definition 5. A space X is called *small-weakly-infinite-dimensional*, abbreviated *small-w.i.d.*, when there is a base $\mathscr{B}$ for X such that $\mathscr{B}^{\text{fin}}$ is inessential. (Here base mains open base.)

In Section 4 we shall discuss other possible definitions and motivate our choice. The aim of this paper is to show that a separable metric space X has a w.i.d. metric compactification cX if and only if X is small-w.i.d.

1. Necessity

Theorem 1. *Let X be a compact space. Then the following statements are equivalent*:

(1) *X is A-w.i.d.*

(2) *X is S-w.i.d.*

(3) *X is small-w.i.d.*

Proof. (1)$\rightarrow$(2). Obvious.

(2)$\rightarrow$(3). Obvious. In fact compactness is not necessary.

(3)$\rightarrow$(1). Let $\{(A_i, B_i)\}_{i=1}^{\infty}$ be an arbitrary sequence of pairs of disjoint closed sets in X. In addition, let $\mathscr{B}$ be a base for X such that $\mathscr{B}^{\text{fin}}$ is inessential. Since X is compact, we can find $U_i^1, U_i^2 \in \mathscr{B}^{\text{fin}}$ containing A_i resp. B_i such that $\bar{U}_i^1 \cap \bar{U}_i^2 = \emptyset$, $i = 1, 2, \ldots$. So we can find open sets V_i, $i = 1, 2, \ldots$, such that

$$A_i \subset U_i^1 \subset V_i \subset \bar{V}_i \subset X - U_i^2 \subset X - B_i$$

and

$$\bigcap_{i=1}^{n} \operatorname{Br} V_i = \emptyset \quad \text{for some } n. \quad \square$$

Proposition 1. *Let X be a separable metric S-w.i.d. space. Moreover, let M be a dense subspace of X.*

Then M is small-w.i.d.

Proof. By virtue of [5, Lemma 3.2] we can construct a countable base $\mathscr{B}$ for X such that for every E, $F \in \mathscr{B}$ we have $\overline{E \cap F} = \bar{E} \cap \bar{F}$. Let $\mathscr{B}_M = \{B \cap M: B \in \mathscr{B}\}$. Then $\mathscr{B}_M$ clearly a base for M. We claim that $\mathscr{B}_M^{\text{fin}}$ is inessential.

Let $\{(B_i^1, B_i^2)\}_{i=1}^{\infty}$ be a sequence of pairs of elements of $\mathscr{B}_M^{\text{fin}}$ such that $B_i^1 \cap B_i^2 = \emptyset$ for every i. Let $B_i^1 = U_i^1 \cap M$ and $B_i^2 = U_i^2 \cap M$, where $U_i^1, U_i^2 \in \mathscr{B}^{\text{fin}}$, for every i. Then clearly $(U_i^1 \cap M) \cap (U_i^2 \cap M) = \emptyset$, from which it follows that $U_i^1 \cap U_i^2 = \emptyset$ (M is dense). Consequently, by our special choice of $\mathscr{B}$, we have $\bar{U}_i^1 \cap \bar{U}_i^2 = \emptyset$.

We conclude that $\{(\bar{U}_i^1, \bar{U}_i^2)\}_{i=1}^{\infty}$ is a sequence of pairs of disjoint closed subsets in X and hence because X is S-w.i.d. we can find open sets V_i for every i such that

$$\bar{U}_i^1 \subset V_i \subset \bar{V}_i \subset X - \bar{U}_i^2 \quad \text{and} \quad \bigcap_{i=1}^{n} \operatorname{Br}_x V_i = \emptyset \quad \text{for some } n.$$

Put $W_i = V_i \cap M$ for every i. Then clearly $B_i^1 \subset W_i \subset \bar{W}_i^m \subset M - B_i^2$ for every i, and $\bigcap_{i=1}^{n} \operatorname{Br}_m W_i = \emptyset$ for some n. We conclude that $\mathscr{B}_M^{\text{fin}}$ is inessential. Hence M is small-w.i.d. □

Theorem 2. *Every separable metric space having a weakly-infinite-dimensional metric compactification is small-w.i.d.*

Proof. Apply Theorem 1 and Proposition 1. □

2. Sufficiency

Theorem 3. *Let X and Y be separable metric spaces such that $X \subset Y$ and X is small-w.i.d. Then there exists a G_δ-set $M \subset Y$ containing X such that M is also small-w.i.d.*

Proof. Without loss of generality we may assume that X is dense in Y.

Let $\mathscr{B}$ be a countable base for X such that $\mathscr{B}^{\text{fin}}$ is inessential. Put $W = \{(B_1, B_2): B_1, B_2 \in \mathscr{B}^{\text{fin}} \text{ and } \bar{B}_1^x \cap \bar{B}_2^x = \emptyset\}$. Put $W(\text{fin})$ the set of all finite subsets of W. Note that $W(\text{fin})$ is countable. Construct a countable collection $\mathscr{V}$ as follows: For every $\{(B_i^1, B_i^2)\}_{i=1}^{n} \in W(\text{fin})$ we do the following:

When there are open V_i, $i = 1, \ldots, n$, such that

$$B_i^1 \subset V_i \subset \bar{V}_i \subset X - B_i^2 \quad \text{and} \quad \bigcap_{i=1}^{n} \operatorname{Br} V_i = \emptyset,$$

select these V_i and put them in $\mathscr{V}$.

By $\overline{}^y$ we shall denote the closure operator in Y. Similarly for Br_y. For every open U in X let $U^y = Y - \overline{(X - U)}^y$. Put $\mathscr{V}^y = \{V^y: V \in \mathscr{V}\}$ and $\mathscr{B}^y = \{B^y: B \in \mathscr{B}\}$, respectively. We see that $V^y \cap X = V$ for every $V \in \mathscr{V}$ and $B^y \cap X = B$ for every $B \in \mathscr{B}$.

Claim. $X \cap \mathrm{Br}_y V^y = \mathrm{Br}\ V$.

Proof of our claim. $\supset$. Follows immediately from the fact that $V^y \cap X = V$.

$\subset$. Let $x \in X \cap \mathrm{Br}_y V^y$ and let U be an open nbd of x in X. From $x \in \mathrm{Br}_y V^y$ we obtain that $U^y \cap V^y \neq \emptyset$. By the fact that X is dense in Y we obtain $U^y \cap V^y \cap X \neq \emptyset$. This leads immediately to $U^y \cap X \cap V^y \cap X \neq \emptyset$ and $U \cap V \neq \emptyset$.

From $x \in \mathrm{Br}_y V^y$ we also obtain that $U^y \cap (X - V^y) \neq \emptyset$. This leads by the definition of V^y to $U^y \cap (X - V)^y \neq \emptyset$. U^y is open so we obtain

$$U^y \cap (X - V) = (U^y \cap X) \cap (X - V) = U \cap (X - V) \neq \emptyset.$$

So U intersects both V and $X - V$ thus $x \in \mathrm{Br}\ V$ and our claim is proved. $\square$

Let ρ be some admissible metric defined on Y. Then, if M is a subset of Y, $\delta(M)$ denotes the diameter of M with respect to the metric ρ. Let

$$G = \bigcap_{n=1}^{\infty} \cup \{B^y \colon \delta(B^y) \leq 1/n,\ B^y \in \mathcal{B}^y\}.$$

Further for $\mathcal{V}^y = \{V_j^y \colon j = 1, 2, \ldots\}$, letting S denote the collection of all finite sets of integers, let

$$H = Y - \left(\bigcup_{s \in S} \left\{ \bigcap_{j \in S} \mathrm{Br}_y V_j^y \colon V_j^y \in \mathcal{V}^y \text{ and } X \cap \bigcap_{j \in S} \mathrm{Br}_y V_j^y = \emptyset \right\} \right).$$

It is clear that G and H are G_δ-sets in Y containing X. So $M = G \cap H$ is a G_δ-set in Y containing X.

Finally we prove that M is small-w.i.d. For this let $\gamma = \{B^y \cap M \colon B^y \in \mathcal{B}^y\}$. Clearly γ is a base for M. We show that γ^{fin} is inessential.

Let $\{(U_i^1, U_i^2)\}_{i=1}^{\infty}$ be a sequence of pairs of elements of γ^{fin} with disjoint closures in M. Let $U_i^l \cap X = B_i^l \in \mathcal{B}^{\mathrm{fin}}$, $l = 1, 2$; then $\overline{B_i^1}^x \cap \overline{B_i^2}^x = \emptyset$. Consider $\{(B_i^1, B_i^2)\}_{i=1}^{\infty}$. Because $\mathcal{B}^{\mathrm{fin}}$ is inessential and our construction of the collection $\mathcal{V}$, we obtain $V_i \in \mathcal{V}$, $i = 1, \ldots, n$, such that

$$B_i^1 \subset V_i \subset \bar{V}_i \subset X - B_i^2 \quad \text{and} \quad \bigcap_{i=1}^{n} \mathrm{Br}\ V_i = \emptyset.$$

Putting $W_i = V_i^y \cap M$ then, by the definition of H, we obtain $\bigcap_{i=1}^n \mathrm{Br}_m W_i = \emptyset$. We also see that $U_i^1 \subset W_i \subset \bar{W}_i^m \subset X - U_i^2$.

Hence we reach to the conclusion that γ^{fin} is inessential. So γ is the required base, which makes our G_δ-extension M of X small-w.i.d. $\square$

Corollary. *Every separable metric small-w.i.d. space X has a small-w.i.d. completion.*

Proposition 2. *Let M be a compact subspace of a small-w.i.d. space X. Then M is small-w.i.d.*

Proof. Let $\mathscr{B}$ be a base for X such that $\mathscr{B}^{\text{fin}}$ is inessential. Let $\mathscr{B}' = \{B \cap M: B \in \mathscr{B}\}$. We shall prove that $\mathscr{B}'^{\text{fin}}$ is inessential in M.

Let $\{(U_i^1, U_i^2)\}_{i=1}^{\infty}$ be a sequence of pairs of elements of $\mathscr{B}'^{\text{fin}}$ with disjoint closures in M. Then $\overline{U_i^1}^m$ is compact, $l = 1, 2$. Clearly we can find $O_i^1, O_i^2 \in \mathscr{B}^{\text{fin}}$ for every i such that

$$\overline{U_i^l}^m \subset O_i^l, \quad l = 1, 2, \quad \text{and} \quad \bar{O}_i^1 \cap \bar{O}_i^2 = \emptyset.$$

For the sequence $\{(O_i^1, O_i^2)\}_{i=1}^{\infty}$ we can find V_i open in X such that

$$O_i^1 \subset V_i \subset \bar{V}_i \subset X - O_i^2 \quad \text{and} \quad \bigcap_{i=1}^{n} \operatorname{Br} V_i = \emptyset \quad \text{for some } n.$$

Putting $W_i = V_i \cap M$ we have

$$U_i^1 \subset W_i \subset \bar{W}_i^m \subset M - U_i^2 \quad \text{and} \quad \bigcap_{i=1}^{n} \operatorname{Br}_m W_i = \emptyset.$$

Hence $\mathscr{B}'^{\text{fin}}$ is inessential and M is small-w.i.d. □

Remark. Later we shall see that every subspace of a separable metric small-w.i.d. space is again small-w.i.d.

The following result is essentially due to R. Pol [6].

Theorem 4. *Let X be a complete separable metric space such that every compact subspace M is w.i.d.*

Then X has a w.i.d. metric compactification cX.

Proof. The space X has a metric compact extension cX such that the remainder $dX = cX - X$ is the countable union of finite dimensional closed sets in cX (see [4, 7]). Then dX is A-w.i.d., being the countable sum of A-w.i.d. subsets. We prove that cX is A-w.i.d.

Let $\{(A_i, B_i)\}_{i=1}^{\infty} = \{(A_{1i}, B_{1i})\}_{i=1}^{\infty} \cup \{(A_{2i}, B_{2i})\}_{i=1}^{\infty}$ be a sequence of disjoint closed sets in cX. Since dX is an A-w.i.d. subset of cX, by virtue of [1, Lemma 1.2.9] we can find open sets U_{1i}, $i = 1, 2, \ldots$, such that

$$A_{1i} \subset U_{1i} \subset \bar{U}_{1i} \subset cX - B_{1i} \quad \text{and} \quad dX \cap \bigcap_{i=1}^{\infty} \operatorname{Br} U_{1i} = \emptyset.$$

Let $M = \bigcap_{i=1}^{\infty} \operatorname{Br} U_{1i}$. Then M is compact and $M \subset X$; hence by our assumptions on X it follows that M is A-w.i.d. So we can find open sets U_{2i}, $i = 1, 2, \ldots$, in cX such that

$$A_{2i} \subset U_{2i} \subset \bar{U}_{2i} \subset cX - B_{2i} \quad \text{and} \quad M \cap \bigcap_{i=1}^{\infty} \operatorname{Br} U_{2i} = \phi.$$

Hence

$$\bigcap_{i=1}^{\infty} \mathrm{Br}\, U_{1i} \cap \bigcap_{i=1}^{\infty} \mathrm{Br}\, U_{2i} = \emptyset. \quad \square$$

Theorem 5. *For a separable metric space X the following statements are equivalent*:

(1) *X is small-w.i.d.*

(2) *X has a small-w.i.d. completion $\hat{X}$.*

(3) *X has a w.i.d. metric compactification cX.*

Proof. (1)$\to$(2). See Theorem 3.

(2)$\to$(3). By Proposition 2 we see that every compact subspace M of $\hat{X}$ is small-w.i.d. Hence, by Theorem 1, M is w.i.d. By this we see that $\hat{X}$ fulfils the conditions for Theorem 4. So $\hat{X}$ has a w.i.d. metric compactification cX.

(3)$\to$(1). See Theorem 2. $\square$

3. Consequences

The space K_ω, consisting of all points in the Hilbert cube I^ω having only finitely many coordinates different from zero, allows no w.i.d. metric compactification, see [3]. Hence K_ω is not small-w.i.d. But K_ω is A-w.i.d.

R. Pol constructed in [6] a complete separable metric space Y such that Y is totally disconnected but not A-w.i.d. He also proved (cf. Theorem 4), that Y has a w.i.d. metric compactification. Hence by Theorem 2 this space is small-w.i.d.

From Theorem 4 and 5 we obtain the following result:

Corollary 1. *For a complete separable metric space X the following statements are equivalent*:

(1) *X is small-w.i.d.*

(2) *Every compact subspace M of X is w.i.d.*

Because every compact subspace of a totally disconnected space is zero-dimensional [1, 1.4.5], we obtain by Theorem 4:

Corollary 2. *Every complete separable metric, totally disconnected space is small-w.i.d.*

In [3] it is announced, see also [2, 4], that every separable metric space having small transfinite dimension has a countable dimensional metric compactification. Because every hereditarily normal c.d. space is A-w.i.d., we obtain by Theorem 2:

Corollary 3. *Every separable metric space having small transfinite dimension is small-w.i.d.*

Finally we have the following:

Corollary 4. *Every subspace M of a separable metric, small-w.i.d. space X is itself small-w.i.d.*

Proof. Let cX be a w.i.d. metric compactification of X, Theorem 5. Then the closure of M in cX is also w.i.d. So again by Theorem 5 the subspace M is small-w.i.d. $\square$

4. Discussion

In this section we want to discuss two other variants of our definition of small-w.i.d.

A question that might arise is the following: Why didn't we take in Definition 4 open sets V_i with

$$\bar{B}_i^1 \subset V_i \subset \bar{V}_i \subset X - \bar{B}_i^2?$$

One can prove that these two definitions of small-w.i.d. are in fact equivalent. Our choice has two reasons:

(i) Our Definition 4 is weaker than the one with the above-mentioned adaptation.

(ii) When for a sequence $\{(B_i^1, B_i^2)\}_{i=1}^{\infty}$ we are searching our required V_i, $i = 1, 2, \ldots$, it can be useful to take $V_i = B_i^1$ or $V_i = X - \bar{B}_i^2$ for some i in certain circumstances.

The second question that could be asked is: Why do we require $\mathscr{B}^{\text{fin}}$ to be inessential instead of $\mathscr{B}$? The reason is that for every separable metric space we can find a base $\mathscr{B}$ such that $\mathscr{B}$ itself is inessential. We give a proof of this fact for the compact metric case for brevity. The proof for the general separable metric case is more complicated, but has the same underlying idea.

Proposition. *Every compact metric space has an inessential base.*

Proof. Let X be a compact metric space and let ρ be an admissible metric on X. Let for a subset M of X, $\delta(M)$ denote the diameter with respect to ρ.

Let $\mathscr{U}$ be a base for X. Then, let $\mathscr{B}_i$ be a finite cover of X of elements of $\mathscr{U}$ such that for every $B \in \mathscr{B}_i$, $\delta(B) \leq 1/i$. Then $\mathscr{B} = \bigcup_{i=1}^{\infty} \mathscr{B}_i$ is a countable base for X and we can write $\mathscr{B}$ as $\mathscr{B} = \{B_j : j = 1, 2, \ldots\}$ such that $\lim_{j\to\infty} \delta(B_j) = 0$. We claim that this base $\mathscr{B}$ is inessential.

Let $\{(B_i^1, B_i^2)\}_{i=1}^{\infty}$ be a sequence of pairs of elements of $\mathscr{B}$ with disjoint closures in X. We distinguish two possible cases.

Case 1. $B_{i_1}^k = B_{i_2}^l$, for some $i_1 \neq i_2$, $k, l \in \{1, 2\}$, $i_1, i_2 \in \{1, 2, \ldots\}$.

Without loss of generality we may assume that $B_{i_1}^1 = B_{i_2}^1$. Then we see that $\bar{B}_{i_1}^1 \cap \overline{(B_{i_1}^2 \cup B_{i_2}^2)} = \emptyset$. Hence we can find open sets V_{i_1}, V_{i_2} such that

$$B_{i_1}^1 = B_{i_2}^1 \subset V_{i_1} \subset \bar{V}_{i_1} \subset V_{i_2} \subset \bar{V}_{i_2} \subset X - (B_{i_1}^2 \cup B_{i_2}^2).$$

Clearly Br $V_{i_1} \cap$ Br $V_{i_2} = \emptyset$. Taking for $i \neq i_1, i_2$, $V_i = B_i^1$, we obtain that, for $i = 1, 2, \ldots$,

$$B_i^1 \subset V_i \subset \bar{V}_i \subset X - B_i^2 \quad \text{and} \quad \bigcap_{i=1}^{\max i_1, i_2} \text{Br } V_i = \emptyset.$$

Case 2. All B_i^k are different for $k = 1, 2$, $i = 1, 2, \ldots$.

Then we have $\lim_{i\to\infty} \delta(B_i^k) = 0$, for $k = 1, 2$. Let $\rho(\bar{B}_1^1, \bar{B}_1^2) = \varepsilon$. Pick j_0 such that $\delta(B_{j_0}^1) < \frac{1}{2}\varepsilon$. Then $\bar{B}_{j_0}^1 \cap \bar{B}_1^1 = \emptyset$ or $\bar{B}_{j_0}^1 \cap \bar{B}_1^2 = \emptyset$. Without loss of generality we may assume that $\bar{B}_{j_0}^1 \cap \bar{B}_1^1 = \emptyset$. Let $V_i = B_i^1$ for all i. Then $B_i^1 \subset V_i \subset \bar{V}_i \subset X - B_i^2$ but also

$$\bigcap_{i=1}^{j_0} \text{Br } V_i \subset \text{Br } V_1 \cap \text{Br } V_{j_0} = \emptyset.$$

Hence both cases show that $\mathcal{B}$ is inessential. □

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